

FAITHFUL REPRESENTATIONS OF MINIMAL DIMENSION OF CURRENT HEISENBERG LIE ALGEBRAS

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ABSTRACT. Given a Lie algebra \mathfrak{g} over a field of characteristic zero k , let $\mu(\mathfrak{g}) = \min\{\dim \pi : \pi \text{ is a faithful representation of } \mathfrak{g}\}$. Let \mathfrak{h}_m be the Heisenberg Lie algebra of dimension $2m + 1$ over k and let $k[t]$ be the polynomial algebra in one variable. Given $m \in \mathbb{N}$ and $p \in k[t]$, let $\mathfrak{h}_{m,p} = \mathfrak{h}_m \otimes k[t]/(p)$ be the current Lie algebra associated to \mathfrak{h}_m and $k[t]/(p)$, where (p) is the principal ideal in $k[t]$ generated by p . In this paper we prove that $\mu(\mathfrak{h}_{m,p}) = m \deg p + \lceil 2\sqrt{\deg p} \rceil$.

1. INTRODUCTION

Let k be a fixed field of characteristic zero. In this paper, all Lie algebras, associative algebras, Hom and tensors are considered over k , unless otherwise is explicitly mentioned.

By Ado's theorem every finite dimensional Lie algebra has a finite dimensional faithful representation (see for instance [J1]). However, given a Lie algebra \mathfrak{g} , it is in general very difficult to compute the number

$$\mu(\mathfrak{g}) = \min\{\dim V : (\pi, V) \text{ is a faithful representation of } \mathfrak{g}\}.$$

The problem of computing the value of $\mu(\mathfrak{g})$, or bounds for it, gained interest since Milnor [Mi] posed the question of which are the finite groups that occur as fundamental groups of complete affinely flat manifolds; in particular whether they are the polycyclic-by-finite groups. There are many articles giving an affirmative answer to Milnor's question under some additional hypothesis, see for instance [Au], [De], [GolK], [GrM], etc. However, the answer to the original Milnor's question is negative in both directions. On the one hand, Margulis [Ma] gave the first complete affinely flat manifold whose fundamental group do not have a polycyclic subgroup of finite index. On the other hand, Benoist [Be] and Burde and Grunewald [BG] found the first examples of nilpotent Lie groups without any left-invariant affine structures. These examples are achieved by finding nilpotent Lie algebras \mathfrak{g} such that $\mu(\mathfrak{g}) > \dim(\mathfrak{g}) + 1$.

Very little is known about the function μ . In particular the value of μ is known only for a few families of Lie algebras, among them, reductive Lie algebras over algebraically closed fields (see [BM]), abelian Lie algebras, and Heisenberg Lie algebras. A brief account of some known results is the following:

1. If \mathfrak{g} is abelian, then $\mu(\mathfrak{g}) = \lceil 2\sqrt{\dim(\mathfrak{g}) - 1} \rceil$. Here $\lceil a \rceil$ is the closest integer that is greater than or equal to a . This result is due to Schur [S], for $k = \mathbb{C}$, and to Jacobson [J2] for arbitrary k (see also [M]).
2. If \mathfrak{h}_m is the Heisenberg Lie algebra of dimension $2m + 1$, then $\mu(\mathfrak{h}_m) = m + 2$, (see [B2]).
3. If \mathfrak{g} is a k -step nilpotent Lie algebra, then $\mu(\mathfrak{g}) \leq 1 + (\dim \mathfrak{g})^k$. This is part of Birkhoffs embedding theorem (see [Bi] and also [R]). In addition, if \mathfrak{g} is \mathbb{Z} -graded, then $\mu(\mathfrak{g}) \leq \dim \mathfrak{g}$ (see [B2]).
4. If \mathfrak{g} is a filiform Lie algebra then $\mu(\mathfrak{g}) \geq \dim \mathfrak{g}$ (see [Be]). The equality holds if $\dim \mathfrak{g} < 10$ (see [B1]).

In this paper we compute the value of μ for a whole family of current Lie algebras associated to Heisenberg Lie algebras. Let $k[t]$ be the polynomial algebra in one variable and given $p \in k[t]$, let (p) denote the principal ideal generated by p in $k[t]$. Given $m \in \mathbb{N}$ and a non-zero polynomial $p \in k[t]$ let

$$\mathfrak{h}_{m,p} = \mathfrak{h}_m \otimes k[t]/(p)$$

be the Lie algebra over k with bracket $[X_1 \otimes q_1, X_2 \otimes q_2] = [X_1, X_2] \otimes q_1 q_2$, $X_i \in \mathfrak{h}_m$, $q_i \in k[t]/(p)$, $i = 1, 2$. It is clear that

$$\dim \mathfrak{h}_{m,p} = (2m + 1) \deg p.$$

The set of Lie algebras $\mathfrak{h}_{m,p}$ constitute a family of 2-step nilpotent Lie algebras that contains the following two subfamilies.

Truncated Heisenberg Lie algebras. This is the subfamily corresponding to $p = t^k$, $k \in \mathbb{N}$. These Lie algebras appear in the literature associated to the Strong Macdonald Conjectures [Mac]. Some articles dealing with these conjectures are [FGT], [HW], [Ku], [T].

Heisenberg Lie algebras over finite extensions of k . This is the subfamily corresponding to polynomials p that are irreducible over k . In this case $\mathfrak{h}_{m,p}$ is the Lie algebra obtained by restricting scalars to k in the Heisenberg Lie algebra over the field $K_p = k[t]/(p)$.

The main result of this paper is the following theorem.

Theorem. *Let $m \in \mathbb{N}$ and $p \in k[t]$, $p \neq 0$. Then*

$$\mu(\mathfrak{h}_{m,p}) = m \deg p + \lceil 2\sqrt{\deg p} \rceil.$$

In order to prove the inequality $\mu(\mathfrak{h}_{m,p}) \geq m \deg p + \lceil 2\sqrt{\deg p} \rceil$, we first prove Theorem 4.4 in which we obtain some fine information about the structure of a faithful representation of an abelian Lie algebra. From Theorem 4.4 it is straightforward to obtain the lower bound part of the Theorem of Schur mentioned above (see Corollary 4.6). One might expect that this result will help to obtain lower bounds of $\mu(\mathfrak{g})$ for other families of nilpotent Lie algebras.

On the other hand, the proof of $\mu(\mathfrak{h}_{m,p}) \leq m \deg p + \lceil 2\sqrt{\deg p} \rceil$ is done by explicitly constructing faithful representations of $\mathfrak{h}_{m,p}$ of minimal dimension.

Example. Let $\mathfrak{h}_m(\mathbb{C})$ be the Heisenberg Lie algebra of dimension $2m+1$ over the complex numbers and let $\mathfrak{h}_m(\mathbb{C})_{\mathbb{R}}$ be $\mathfrak{h}_m(\mathbb{C})$ viewed as a Lie algebra over \mathbb{R} . We know that $\mu(\mathfrak{h}_m(\mathbb{C})) = m+2$ and the faithful representation of $\mathfrak{h}_m(\mathbb{C})$

in \mathbb{C}^{m+2} yields a faithful representation of $\mathfrak{h}_m(\mathbb{C})_{\mathbb{R}}$ of dimension $2m + 4$. However, from the above theorem we obtain that $\mu(\mathfrak{h}_m(\mathbb{C}))_{\mathbb{R}} = 2m + 3$. If $\{X, Y, Z\}$ is the basis of $\mathfrak{h}_1(\mathbb{C})$, with $[X, Y] = Z$, then

$$\pi((x_1 + ix_2)X + (y_1 + iy_2)Y + (z_1 + iz_2)Z) = \begin{pmatrix} 0 & x_1 & x_2 & z_1 & z_2 \\ 0 & 0 & 0 & y_1 & y_2 \\ 0 & 0 & 0 & -y_2 & y_1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

is a faithful representation of $\mathfrak{h}_1(\mathbb{C})_{\mathbb{R}}$ in \mathbb{R}^5 .

2. PRELIMINARIES

Given a finite dimensional vector space V , a representation of an associative algebra A on V is an associative algebra homomorphism $\pi : A \rightarrow \text{End}(V)$ and similarly, a representation of a Lie algebra \mathfrak{g} on V is Lie algebra homomorphism $\pi : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$. A *nil-representation* is, by definition, a representation whose image is contained in the set of nilpotent endomorphisms. All representations (π, V) considered in this paper will be finite dimensional. Thus, given a basis of V , we can express each operator $\pi(X)$ by its associated matrix. When the basis is fixed we shall denote this matrix also by $\pi(X)$. The space of rectangular matrices of size $m \times n$ with entries in k will be denoted by $M_{m,n}(k)$.

A representation is *faithful* if it is an injective homomorphism. Considering the left action, it is clear that any associative algebra with unit of dimension n has a faithful representation of dimension n . On the other hand, a theorem due to Ado states that any finite dimensional Lie algebra has a finite dimensional faithful representation (see for instance [J1]). Given a Lie algebra \mathfrak{g} , let

$$\mu(\mathfrak{g}) = \min\{\dim V : (\pi, V) \text{ is a faithful representation of } \mathfrak{g}\}.$$

Given a Lie algebra \mathfrak{g} and a commutative associative algebra A the tensor product $\mathfrak{g} \otimes A$ has a Lie algebra structure with bracket

$$[X_1 \otimes a_1, X_2 \otimes a_2] = [X_1, X_2] \otimes a_1 a_2,$$

$X_i \in \mathfrak{g}$, $a_i \in A$, $i = 1, 2$. This Lie algebra is known as the *current Lie algebra* associated to \mathfrak{g} and A (see for instance [GoR], [Z]).

Remark 2.1. Note that $\mathfrak{g} \otimes A$ could be viewed as a Lie algebra over the algebra A but, as we have already mentioned, we look at it as a Lie algebra over k . Occasionally we shall make an exception when $A = K$ is a field extension of k . In this case, we shall denote by \mathfrak{g}_K the Lie algebra $\mathfrak{g} \otimes K$ whenever it is viewed as a Lie algebra over K .

If (π, V) is a representation of a Lie algebra \mathfrak{g} and (ρ, W) is a representation of a commutative associative algebra A , then it is clear that $\pi \otimes \rho : \mathfrak{g} \otimes A \rightarrow \mathfrak{gl}(V \otimes W)$, given by $(\pi \otimes \rho)(X \otimes a) = \pi(X) \otimes \rho(a)$, is a representation of the Lie algebra $\mathfrak{g} \otimes A$. It is clear that if π and ρ are injective, then so it is $\pi \otimes \rho$. Therefore, if A has an identity, by considering the regular representation of A we obtain that

$$(2.1) \quad \mu(\mathfrak{g} \otimes A) \leq \mu(\mathfrak{g}) \dim A.$$

If $A = K$ is a field extension of k and we look at $\mathfrak{g}_K = \mathfrak{g} \otimes K$ as a Lie algebra over K , then the above construction also yields a K -representation of \mathfrak{g}_K and in this case one has

$$(2.2) \quad \mu(\mathfrak{g}_K) \leq \mu(\mathfrak{g}).$$

The Heisenberg Lie algebra \mathfrak{h}_m is the k -vector space of dimension $2m + 1$ with a basis $\{X_1, \dots, X_m, Y_1, \dots, Y_m, Z\}$ whose only non-zero brackets are

$$[X_i, Y_i] = Z.$$

It is clear that the center of \mathfrak{h}_m is $\mathfrak{z}(\mathfrak{h}_m) = kZ$. This Lie algebra has a well known faithful representation (π_0, k^{m+2}) that, in terms of the canonical basis of k^{m+2} , is given by

$$(2.3) \quad \pi_0\left(\sum_{i=1}^m x_i X_i + \sum_{i=1}^m y_i Y_i + zZ\right) = \begin{pmatrix} 0 & x_1 & \dots & x_m & z \\ & & & & y_1 \\ & & & & \vdots \\ & & & 0 & y_m \\ & & & & 0 \end{pmatrix}, \quad x_i, y_i, z \in k.$$

It is known that any other faithful representation of \mathfrak{h}_m has dimension greater than or equal to $m + 2$ (see [B2]) and thus $\mu(\mathfrak{h}_m) = m + 2$.

Let $k[t]$ be the polynomial algebra in one variable, let $p = a_0 + \dots + a_{d-1}t^{d-1} + t^d$ be a non-zero monic polynomial and let (p) be the principal ideal generated by p . The regular representation ρ of the quotient algebra $k[t]/(p)$ is expressed, in terms of the canonical basis $\{1, t, \dots, t^{d-1}\}$, by $\rho(t^i) = P^i$, where

$$(2.4) \quad P = \begin{pmatrix} 0 & & & -a_0 \\ 1 & 0 & & -a_1 \\ & 1 & \ddots & \vdots \\ & & \ddots & 0 \\ & & & 1 & -a_{d-1} \end{pmatrix}$$

is the matrix associated to p .

For any $m \in \mathbb{N}$ and any non-zero polynomial $p \in k[t]$ let

$$\mathfrak{h}_{m,p} = \mathfrak{h}_m \otimes k[t]/(p)$$

be the *current Heisenberg Lie algebra* associated to m and p . Note that (2.1) yields

$$\mu(\mathfrak{h}_{m,p}) \leq (m + 2) \deg p = m \deg p + 2 \deg p.$$

The main goal of this paper is to prove that in fact

$$\mu(\mathfrak{h}_{m,p}) = m \deg p + \left\lceil 2\sqrt{\deg p} \right\rceil.$$

Note that this extends the known result $\mu(\mathfrak{h}_m) = m + 2$.

We now recall and prove some results, needed in the following sections, about finite dimensional representations of nilpotent Lie algebras.

Let \mathfrak{n} be finite dimensional nilpotent Lie algebra and let (π, V) be a finite dimensional representation of \mathfrak{n} . If k is algebraically closed, a well known theorem of Zassenhaus (see [J1]) states that V can be decomposed as

$$V = V_1 \oplus V_2 \oplus \dots \oplus V_r,$$

so that, for all $X \in \mathfrak{n}$ and $i = 1, \dots, r$, $\pi(X)|_{V_i}$ is an scalar $\lambda_i(X)$ plus a nilpotent operator $N_i(X)$ on V_i .

A consequence of this result is that every representation of a nilpotent Lie algebra has a Jordan decomposition.

Theorem 2.2. *Let \mathfrak{n} be finite dimensional nilpotent Lie algebra and let (π, V) be a finite dimensional representation of \mathfrak{n} . For each $X \in \mathfrak{n}$ let $\pi_S(X)$ and $\pi_N(X)$ be, respectively, the semisimple and nilpotent part of the additive Jordan decomposition of $\pi(X)$. Then (π_S, V) and (π_N, V) are representations of \mathfrak{n} . Moreover, $\pi_S(X) = 0$ for all $X \in [\mathfrak{n}, \mathfrak{n}]$.*

Proof. We first assume that k is algebraically closed. By Zassenhaus' theorem

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_r,$$

and $\pi(X)|_{V_i}$ is an scalar $\lambda_i(X)$ plus a nilpotent operator $N_i(X)$ on V_i , for all $X \in \mathfrak{n}$ and $i = 1, \dots, r$. Since $\lambda_i(X) = \text{trace}(\pi(X)|_{V_i}) / \dim(V_i)$ it follows that λ_i is linear and $\lambda_i(X) = 0$ for all $X \in [\mathfrak{n}, \mathfrak{n}]$. Hence λ_i is a Lie algebra homomorphism. Moreover, since $N_i = \pi|_{V_i} - \lambda_i$ and $\lambda_i(X)$ is a scalar, it follows that N_i is a Lie algebra homomorphism. Since $\pi_S|_{V_i} = \lambda_i$ and $\pi_N|_{V_i} = N_i$ the theorem follows for k algebraically closed.

Now assume k is arbitrary (with $\text{char}(k) = 0$). Let \bar{k} be the algebraic closure of k , $\bar{\mathfrak{g}} = \mathfrak{g} \otimes_k \bar{k}$, $\bar{V} = V \otimes_k \bar{k}$ and $\bar{\pi} = \pi \otimes_k 1$. We know that $\bar{\pi}_S$ and $\bar{\pi}_N$ are Lie algebras homomorphisms and we must prove that π_S and π_N are Lie algebras homomorphisms.

Let $B = \{v_1, \dots, v_n\}$ be a basis of V , $\bar{B} = \{v_1 \otimes 1, \dots, v_n \otimes 1\}$ the corresponding basis of \bar{V} and, given an operator T on V (resp. on \bar{V}) let $[T]_B$ (resp. $[\bar{T}]_{\bar{B}}$) be the associated matrix with respect to the basis B (resp. \bar{B}). Since $[\pi(X)]_B = [\bar{\pi}(X \otimes_k 1)]_{\bar{B}}$ for all $X \in \mathfrak{g}$, it follows that

$$(2.5) \quad [\bar{\pi}(X \otimes_k 1)]_{\bar{B}} = [\pi_S(X)]_B + [\pi_N(X)]_B.$$

Since $[\pi_S(X)]_B$ and $[\pi_N(X)]_B$ are respectively semisimple and nilpotent matrices that commute, it follows that (2.5) is the Jordan decomposition of the matrix $[\bar{\pi}(X \otimes_k 1)]_{\bar{B}}$, that is $[\pi_S(X)]_B = [\bar{\pi}_S(X \otimes_k 1)]_{\bar{B}}$ and $[\pi_N(X)]_B = [\bar{\pi}_N(X \otimes_k 1)]_{\bar{B}}$. Therefore π_S and π_N are Lie algebras homomorphisms. \square

Definition 2.3. Let (π, V) be a finite dimensional representation of a finite dimensional nilpotent Lie algebra \mathfrak{n} . We call the representations (π_S, V) and (π_N, V) the *semisimple part* and the *nilpotent part* of (π, V) respectively.

For certain nilpotent Lie algebras it is enough to consider nilrepresentations in the definition of μ as the following theorem shows.

Theorem 2.4. *Let \mathfrak{n} be a finite dimensional nilpotent Lie algebra such that the center $\mathfrak{z}(\mathfrak{n})$ is contained in $[\mathfrak{n}, \mathfrak{n}]$, and let (π, V) be a finite dimensional representation of \mathfrak{n} . Then (π, V) is faithful if and only if (π_N, V) is faithful.*

Proof. We only need to show that if π is injective, then so it is π_N . Let us assume that π is injective and let $X_0 \in \mathfrak{n}$ such that $\pi_N(X_0) = 0$. Since $\pi_S|_{[\mathfrak{n}, \mathfrak{n}]} = 0$ it follows that

$$\begin{aligned} \pi([X_0, X]) &= \pi_N([X_0, X]) + \pi_S([X_0, X]) \\ &= [\pi_N(X_0), \pi_N(X)] \\ &= 0 \end{aligned}$$

for all $X \in \mathfrak{n}$. Since π is injective we obtain that $X_0 \in \mathfrak{z}(\mathfrak{n}) \subset [\mathfrak{n}, \mathfrak{n}]$. Finally, since $\pi|_{[\mathfrak{n}, \mathfrak{n}]} = \pi_N|_{[\mathfrak{n}, \mathfrak{n}]}$ it follows that $\pi(X_0) = 0$ and therefore $X_0 = 0$. \square

3. A FAMILY OF REPRESENTATIONS OF $\mathfrak{h}_{m,p}$ AND THE UPPER BOUND FOR $\mu(\mathfrak{h}_{m,p})$

Let $m \in \mathbb{N}$ be a fixed natural number and let $p = a_0 + \cdots + a_{d-1}t^{d-1} + t^d$ be a fixed non-zero monic polynomial of degree d . If $(\pi_0, \mathfrak{k}^{m+2})$ is the faithful representation of \mathfrak{h}_m defined in (2.3) and $(\rho, \mathfrak{k}[t]/(p))$ is the regular representation of $\mathfrak{k}[t]/(p)$, then the representation

$$\pi_0 \otimes \rho : \mathfrak{h}_{m,p} \rightarrow \mathfrak{gl}(\mathfrak{k}^{m+2} \otimes \mathfrak{k}[t]/(p))$$

is expressed, in terms of the canonical basis of the tensor product $\mathfrak{k}^{m+2} \otimes \mathfrak{k}[t]/(p)$, by the blocked matrix

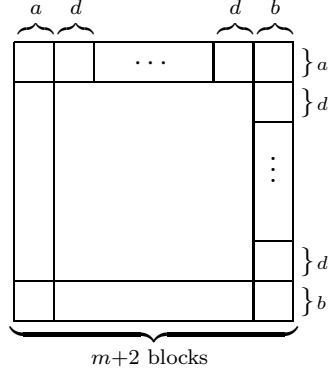
$$\begin{aligned} (\pi_0 \otimes \rho) \left(\sum_{i=1}^m X_i \otimes q_{1,i}(t) + \sum_{i=1}^m Y_i \otimes q_{2,i}(t) + Z \otimes q_3(t) \right) \\ = \begin{pmatrix} 0 & q_{1,1}(P) & \cdots & q_{1,m}(P) & q_3(P) \\ & & & & q_{2,1}(P) \\ & & & 0 & \vdots \\ & & & & q_{2,m}(P) \\ & & & & 0 \end{pmatrix} \end{aligned}$$

of size $(m+2)d$. We shall now construct a family of representations of $\mathfrak{h}_{m,p}$ that contains $\pi_0 \otimes \rho$.

Definition 3.1. Given two natural numbers a and b and two matrices $A \in M_{a,d}(\mathfrak{k})$ and $B \in M_{d,b}(\mathfrak{k})$, let $(\pi_{A,B}, \mathfrak{k}^{md+a+b})$ be the representation of $\mathfrak{h}_{m,p}$ that, in terms of the canonical basis of \mathfrak{k}^{md+a+b} , is given by the blocked matrix

$$\begin{aligned} \pi_{A,B} \left(\sum_{i=1}^m X_i \otimes q_{1,i}(t) + \sum_{i=1}^m Y_i \otimes q_{2,i}(t) + Z \otimes q_3(t) \right) \\ = \begin{pmatrix} 0 & A q_{1,1}(P) & \cdots & A q_{1,m}(P) & A q_3(P) B \\ & & & & q_{2,1}(P) B \\ & & & 0 & \vdots \\ & & & & q_{2,m}(P) B \\ & & & & 0 \end{pmatrix}, \end{aligned}$$

whose block structure is depicted below:



The following facts are not difficult to prove:

- (1) $(\pi_{A,B}, \mathbb{k}^{md+a+b})$ is a representation of $\mathfrak{h}_{m,p}$ for any A and B .
- (2) If $a = b = d$ and A and B are the identity matrix, then $\pi_{A,B} = \pi_0 \otimes \rho$.
- (3) Let $\mathbb{k}[P] \subset M_{d,d}(\mathbb{k})$ be the subalgebra generated by P in $M_{d,d}(\mathbb{k})$ and let $\beta_{A,B} : \mathbb{k}[P] \rightarrow M_{a,b}(\mathbb{k})$ be the linear map given by $\beta_{A,B}(q(P)) = Aq(P)B$. Then $\pi_{A,B}$ is injective if and only if $\beta_{A,B}$ is injective.

Now the problem of finding the minimal possible dimension for a faithful representation among the family $\pi_{A,B}$ reduces to finding the minimal value of $a + b$ among the pairs (a, b) for which there exists two matrices $A \in M_{a,d}(\mathbb{k})$ and $B \in M_{d,b}(\mathbb{k})$ such that $\beta_{A,B}$ is injective. Since $\dim(\mathbb{k}[P]) = d$ and $\dim(M_{a,b}(\mathbb{k})) = ab$ it follows necessarily that $ab \geq d$. We shall see now that this condition is sufficient as well. Let $A \in M_{a,d}$ and $B \in M_{d,b}$ be the following matrices

$$(3.1) \quad A_{ij} = \begin{cases} 1, & \text{if } j = d - (a - i)b; \\ 0, & \text{otherwise;} \end{cases} \quad B_{ij} = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{otherwise.} \end{cases}$$

$$\text{That is } B = \left(\underbrace{\begin{pmatrix} 1 & & 0 & \dots & 0 \\ & \ddots & \vdots & \ddots & \vdots \\ & & 0 & \dots & 0 \end{pmatrix}}_d, \underbrace{\begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}}_{b-d} \right), \text{ if } b \geq d; \quad B = \left(\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ 0 & \dots & 0 & \\ \vdots & \ddots & \vdots & \\ 0 & \dots & 0 & \end{pmatrix} \right) \begin{matrix} \left. \vphantom{\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ 0 & \dots & 0 & \\ \vdots & \ddots & \vdots & \\ 0 & \dots & 0 & \end{pmatrix}} \right\}^b \\ \left. \vphantom{\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ 0 & \dots & 0 & \\ \vdots & \ddots & \vdots & \\ 0 & \dots & 0 & \end{pmatrix}} \right\}^{d-b} \end{matrix} \quad \text{if } d \geq b$$

$$\text{and, for instance, if } d = 6, a = 4 \text{ and } b = 2 \text{ we have } A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Theorem 3.2. *If $ab \geq d$ and A and B are the matrices defined above, then $\beta_{A,B}$ is injective.*

Proof. Let $q \in \mathbb{k}[t]$ be a monic polynomial such that $\deg(q) < d$ and let us show that $Aq(P)B \neq 0$. It is easy to prove by induction that, for $1 \leq j \leq d - k$, one has

$$(P^k)_{ij} = \begin{cases} 1, & \text{if } i = j + k; \\ 0, & \text{if } i \neq j + k. \end{cases}$$

This implies that, for $1 \leq j \leq d - k$, one has

$$(P^k B)_{ij} = \begin{cases} 1, & \text{if } i = j + k; \\ 0, & \text{if } i \neq j + k; \end{cases}$$

and therefore, for $1 \leq j \leq d - k$, one has

$$(AP^k B)_{ij} = \begin{cases} 1 & \text{if } j + k = d - (a - i)b; \\ 0 & \text{if } j + k \neq d - (a - i)b. \end{cases}$$

Let $k_0 = \deg(q)$ and let us assume for a moment that there exist i_0 and j_0 such that

1. $1 \leq j_0 \leq \min\{b, d - k_0\}$,
2. $1 \leq i_0 \leq a$ and
3. $j_0 + k_0 = d - (a - i_0)b$.

Now we have

$$(AP^k B)_{i_0, j_0} = \begin{cases} 1, & \text{if } k = k_0; \\ 0, & \text{if } k = 0, \dots, k_0 - 1; \end{cases}$$

and this implies that $(Aq(P)B)_{i_0, j_0} = 1$ and thus $Aq(P)B \neq 0$ as we wanted to prove.

In order to prove the existence of i_0 and j_0 satisfying properties 1–3 above, we shall see that $t_0 = \left\lfloor \frac{d - k_0}{b} - 1 \right\rfloor$ satisfies $0 \leq t_0 \leq a - 1$ and $1 \leq d - k_0 - t_0 b \leq b$. Once this is proved, $i_0 = a - t_0$ and $j_0 = d - k_0 - t_0 b$ have the desired properties.

It is clear that $t_0 \geq 0$. Since $k_0 < d \leq ab$ it follows that $\left\lfloor \frac{d - k_0 - 1}{b} \right\rfloor \leq a - 1$. It is easy to see that $\left\lceil \frac{x}{y} - 1 \right\rceil \leq \left\lfloor \frac{x - 1}{y} \right\rfloor$ for all $x, y \in \mathbb{N}$, in particular

$$t_0 = \left\lceil \frac{d - k_0}{b} - 1 \right\rceil \leq \left\lfloor \frac{d - k_0 - 1}{b} \right\rfloor \leq a - 1.$$

Additionally

$$\frac{d - k_0}{b} - 1 \leq t_0 \leq \frac{d - k_0 - 1}{b}$$

and thus $b \geq d - k_0 - t_0 b \geq 1$. □

Corollary 3.3. *Let $p \in \mathbb{k}[t]$ be non-zero polynomial. Then, for all $m \in \mathbb{N}$*

$$\mu(\mathfrak{h}_{m,p}) \leq m \deg(p) + \left\lceil 2\sqrt{\deg(p)} \right\rceil.$$

Proof. By Proposition 3.2, $\mu(\mathfrak{h}_{m,p}) \leq m \deg(p) + a + b$ for all a and b such that $ab > \deg(p)$. Since

$$(3.2) \quad \min\{a + b : a, b \in \mathbb{N} \text{ and } ab \geq d\} = \left\lceil 2\sqrt{d} \right\rceil$$

for all $d \in \mathbb{N}$, we obtain the desired inequality. □

4. THE LOWER BOUND FOR $\mu(\mathfrak{h}_{m,p})$

From the inequality (2.2) we know that

$$\mu\left(\left(\mathfrak{g} \otimes k[t]/(p)\right)_K\right) \leq \mu(\mathfrak{g} \otimes k[t]/(p)).$$

Since $(\mathfrak{g} \otimes k[t]/(p))_K \simeq \mathfrak{g}_K \otimes_K K[t]/(p)$ as Lie algebras over K , we obtain

$$\mu(\mathfrak{g}_K \otimes_K K[t]/(p)) \leq \mu(\mathfrak{g} \otimes k[t]/(p)).$$

Therefore, in order to obtain a lower bound for $\mu(\mathfrak{h}_{m,p})$ we may assume that k is algebraically closed. In this case $p = (t - b_1)^{d_1} \dots (t - b_r)^{d_r}$ for different $b_l \in k$ and

$$\begin{aligned} \mathfrak{h}_{m,p} &\simeq \bigoplus_{l=1}^r \mathfrak{h}_m \otimes k[t]/((t - b_l)^{d_l}) \\ &\simeq \bigoplus_{l=1}^r \mathfrak{h}_{m,t^{d_l}}. \end{aligned}$$

This section is devoted to prove the following theorem.

Theorem 4.1. *If (π, V) is a faithful representation of $\bigoplus_{l=1}^r \mathfrak{h}_{m,t^{d_l}}$ and $d = \sum_{l=1}^r d_l$, then*

$$\dim V \geq md + \left\lceil 2\sqrt{d} \right\rceil.$$

In particular $\mu(\mathfrak{h}_{m,p}) \geq m \deg p + \left\lceil 2\sqrt{\deg p} \right\rceil$.

In what follows we shall assume that the exponents d_1, \dots, d_r are fixed and we set $d = \sum_{l=1}^r d_l$.

Let us define the following elements in $\bigoplus_{l=1}^r \mathfrak{h}_{m,t^{d_l}}$:

$$\begin{aligned} X_{i,l}^j &= (0, \dots, 0, X_i \otimes t^j, 0, \dots, 0), \\ Y_{i,l}^j &= (0, \dots, 0, Y_i \otimes t^j, 0, \dots, 0), \\ Z_l^j &= (0, \dots, 0, Z \otimes t^j, 0, \dots, 0), \end{aligned}$$

where the non-zero component of each element is in the l^{th} coordinate. Thus the set

$$\left\{ X_{i,l}^j, Y_{i,l}^j, Z_l^j : 1 \leq i \leq m, 0 \leq j \leq d_l - 1, 1 \leq l \leq r \right\}$$

is a basis of $\bigoplus_{l=1}^r \mathfrak{h}_{m,t^{d_l}}$. The center \mathfrak{z} of $\bigoplus_{l=1}^r \mathfrak{h}_{m,t^{d_l}}$ is spanned by the set $\{Z_l^j : 0 \leq j \leq d_l - 1, 1 \leq l \leq r\}$.

Lemma 4.2. *For any $X \in \bigoplus_{l=1}^r \mathfrak{h}_{m,t^{d_l}}$ such that $X \notin \mathfrak{z}$ there exist $Y \in \bigoplus_{l=1}^r \mathfrak{h}_{m,t^{d_l}}$ such that $[X, Y] = Z_l^{d_l-1}$ for some $l = 1, \dots, r$.*

Proof. Assume that

$$X = \sum_{l=1}^r \sum_{j=0}^{d_l-1} \sum_{i=1}^m a_{i,j,l} X_{i,l}^j + b_{i,j,l} Y_{i,l}^j + c_{j,l} Z_l^j$$

for some $a_{i,j,l}, b_{i,j,l}, c_{j,l} \in k$. Since X is not in the center of $\bigoplus_{l=1}^r \mathfrak{h}_{m,t^{d_l}}$, then there exists some (i_0, j_0, l_0) such that either $a_{i_0,j_0,l_0} \neq 0$ or $b_{i_0,j_0,l_0} \neq 0$.

Assuming that $a_{i_0, j_0, l_0} \neq 0$, let $j_1 = \min\{j : a_{i_0, j, l_0} \neq 0\}$ and let $Y = \frac{1}{a_{i_0, j_1, l_0}} Y_{i_0, l_0}^{d_{i_0}-1-j_1}$ then $[X, Y] = Z_{l_0}^{d_{i_0}-1}$. If $b_{i_0, j_0, l_0} \neq 0$ the argument is analogous. \square

Lemma 4.3. *Let \mathfrak{g} be a Lie subalgebra of $\bigoplus_{l=1}^r \mathfrak{h}_{m, t^{d_l}}$ such that $Z_l^{d_l-1} \notin \mathfrak{g}$ for all $l = 1, \dots, r$. Then*

$$\dim \mathfrak{g} \leq md + \dim \mathfrak{g} \cap \mathfrak{z}.$$

Proof. Let $\mathfrak{z}_0 = \mathfrak{g} \cap \mathfrak{z}$. Since by hypothesis $Z_l^{d_l-1} \notin \mathfrak{z}_0$ for all $l = 1, \dots, r$, we may choose a linear functional $\alpha : \mathfrak{z} \rightarrow \mathbb{k}$ such that $\alpha|_{\mathfrak{z}_0} = 0$ and $\alpha(Z_l^{d_l-1}) \neq 0$ for all $l = 1, \dots, r$.

Let \mathfrak{g}_0 be a complementary subspace of \mathfrak{z}_0 in \mathfrak{g} let $\tilde{\mathfrak{z}}$ be a complementary subspace of \mathfrak{z}_0 in \mathfrak{z} , and let $\tilde{\mathfrak{g}}$ be a complementary subspace of $\mathfrak{g} \oplus \tilde{\mathfrak{z}}$ in $\bigoplus_{l=1}^r \mathfrak{h}_{m, d_l}$. Thus $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{z}_0$, $\mathfrak{z} = \mathfrak{z}_0 \oplus \tilde{\mathfrak{z}}$ and

$$(4.1) \quad \bigoplus_{l=1}^r \mathfrak{h}_{m, t^{d_l}} = \tilde{\mathfrak{g}} \oplus \mathfrak{g}_0 \oplus \mathfrak{z}_0 \oplus \tilde{\mathfrak{z}}$$

as vector spaces. Let $V = \tilde{\mathfrak{g}} \oplus \mathfrak{g}_0$ and let

$$B : V \times V \rightarrow \mathbb{k} \\ (X, Y) \mapsto \alpha([X, Y]).$$

It is clear that B is an skew-symmetric bilinear form on V .

Let us prove that B is nondegenerate. Given $X \in V$, $X \neq 0$, we know from Lemma 4.2 that there exist $Y \in \bigoplus_{l=1}^r \mathfrak{h}_{m, t^{d_l}}$ such that $[X, Y] = Z_l^{d_l-1}$ for some l . If $\tilde{Y} \in V$ is the projection of Y to V with respect to the decomposition (4.1), then $B(X, \tilde{Y}) \neq 0$.

Let us see now that \mathfrak{g}_0 is a B -isotropic subspace. If $X, Y \in \mathfrak{g}_0$ then, since \mathfrak{g} is a Lie subalgebra, it follows that $[X, Y] \in \mathfrak{z}_0$, and since $\alpha|_{\mathfrak{z}_0} = 0$ we obtain that $B(X, Y) = 0$.

Since B is a nondegenerate bilinear form on V and \mathfrak{g}_0 is a B -isotropic subspace of V , it follows that $\dim \mathfrak{g}_0 \leq \frac{\dim V}{2} = md$ and therefore $\dim \mathfrak{g} \leq md + \dim \mathfrak{z}_0$. \square

In order to prove Theorem 4.1 we need the following result that gives some precise information about the structure of a commuting family of nilpotent operators on a vector space.

Theorem 4.4. *Let V be a finite dimensional vector space and let \mathcal{N} be a non-zero abelian subspace of $\text{End}(V)$ consisting of nilpotent operators. Then there exist a linearly independent set $B = \{v_1, \dots, v_s\} \subset V$ and a decomposition $\mathcal{N} = \mathcal{N}_1 \oplus \dots \oplus \mathcal{N}_s$, with $\mathcal{N}_i \neq 0$ for all i , such that the applications $F_i : \mathcal{N} \rightarrow V$ defined by $F_i(N) = N(v_i)$ satisfy*

- (1) $F_i|_{\mathcal{N}_i}$ is injective for all $i = 1 \dots s$;
- (2) $\mathcal{N}_j \subset \ker F_i$ for all $1 \leq i < j \leq s$;
- (3) $\mathcal{N}_j V \subset \text{im } F_i|_{\mathcal{N}_i}$ for all $1 \leq i < j \leq s$.

Furthermore, given a finite subset $\{N_1, \dots, N_q\}$ of non-zero operators in \mathcal{N} , the vector v_1 can be chosen so that $N_k(v_1) \neq 0$ for all $k = 1, \dots, q$.

We first prove the following lemma.

Lemma 4.5. *Let V be a finite dimensional vector space over k , let \mathcal{F} a non-zero subspace of $\text{End}(V)$ and let $r = \max\{\dim \mathcal{F}v : v \in V\}$. Then for any finite subset $\{T_1, T_2, \dots, T_q\} \subseteq \mathcal{F}$, such that $T_i \neq 0$ for all $i = 1, \dots, q$, there exist $v_0 \in V$ such that $r = \dim \mathcal{F}v_0$ and $T_i(v_0) \neq 0$ for all $i = 1, \dots, q$.*

Proof. We will prove the lemma by induction on q . It is clear that the lemma is true in the case $q = 0$. Let $\{T_1, \dots, T_q, T_{q+1}\} \subseteq \mathcal{F}$, by inductive hypothesis there exist $v'_0 \in V$ such that $r = \dim \mathcal{F}v'_0$ and $T_i(v'_0) \neq 0$ for all $i = 1, \dots, q$. If $T_{q+1}(v'_0) \neq 0$ we take $v_0 = v'_0$.

Suppose that $T_{q+1}(v'_0) = 0$. Since $T_{q+1} \neq 0$ there exist $w \in V$ such that $T_{q+1}(w) \neq 0$. Let us take $\tilde{T}_1, \dots, \tilde{T}_r \in \mathcal{F}$ such that $\{\tilde{T}_1(v'_0), \dots, \tilde{T}_r(v'_0)\}$ is a basis of $\mathcal{F}v'_0$.

We now claim that there exists t_0 such that

- (1) the set $\{\tilde{T}_1(v'_0 + t_0w), \dots, \tilde{T}_r(v'_0 + t_0w)\}$ is linearly independent, and
- (2) $T_i(v'_0 + t_0w) \neq 0$ for all $i = 1, \dots, q + 1$.

In order to prove this fact, let B be a basis of V . Let A_t be the matrix whose columns are the coordinates of the vectors $\tilde{T}_1(v'_0 + tw), \dots, \tilde{T}_r(v'_0 + tw)$ and let $a(t)$ be the $r \times r$ minor of A_t such that $a(0) \neq 0$ (since $\{\tilde{T}_1(v'_0), \dots, \tilde{T}_r(v'_0)\}$ is linearly independent the existence of this minor is granted). For $i = 1, \dots, q$, let $p_i(t)$ be a coordinate of $T_i(v'_0 + tw)$ such that $p_i(0) \neq 0$ and let $p_{q+1}(t)$ be a coordinate of $T_{q+1}(v'_0 + tw)$ such that $p_{q+1}(1) \neq 0$ (recall that we assumed that $T_{q+1}(v'_0) = 0$). Now $\{a(t), p_1(t), \dots, p_{q+1}(t)\}$ is a finite set of non-zero polynomials and thus there exist t_0 such that non of them vanish at t_0 . For this t_0 conditions (1) and (2) are verified and taking $v_0 = v'_0 + t_0w$ we complete the inductive argument. \square

Proof of Theorem 4.4. We shall proceed by induction on $\dim \mathcal{N}$. If $\dim \mathcal{N} = 1$, let N_0 be any non-zero operator in \mathcal{N} and let v_1 be a vector such that $N_0(v_1) \neq 0$. If we take $B = \{v_1\}$ and $\mathcal{N}_1 = \mathcal{N}$ then $F_1|_{\mathcal{N}_1}$ is injective and conditions (2) and (3) are empty. It is clear that if N_1, \dots, N_q are non-zero operators in \mathcal{N} then $N_k(v_1) \neq 0$ for all $k = 1, \dots, q$.

Now assume that the theorem is true for any non-zero abelian subspace of $\text{End}(V)$ of dimension less than $\dim \mathcal{N}$.

Let $r = \max\{\dim \mathcal{N}v : v \in V\} > 0$. By Lemma 4.5, there exist $v_1 \in V$ such that $r = \dim \mathcal{N}v_1$ and $N_k(v_1) \neq 0$ for all $k = 1, \dots, q$. Let $F_1 : \mathcal{N} \rightarrow V$ defined by $F_1(N) = N(v_1)$. If F_1 is injective then we take $B = \{v_1\}$ and $\mathcal{N}_1 = \mathcal{N}$. We obtain that $F_1|_{\mathcal{N}_1}$ is injective and since conditions (2) and (3) are empty, we are done.

Otherwise, let $\mathcal{N}' = \ker F_1$. Since $r = \dim \mathcal{N}v_1 > 0$, we have $\dim \mathcal{N}' < \dim \mathcal{N}$. By the inductive hypothesis, there exist a linearly independent set $B' = \{v_2, \dots, v_s\} \subset V$ and a decomposition $\mathcal{N}' = \mathcal{N}_2 \oplus \dots \oplus \mathcal{N}_s$, with $\mathcal{N}_i \neq 0$ for $i = 2, \dots, s$, such that

- (1') $F_i|_{\mathcal{N}_i}$ is injective for all $i = 2 \dots s$;
- (2') $\mathcal{N}_j \subset \ker F_i$ for all $2 \leq i < j \leq s$;
- (3') $\mathcal{N}_j V \subset \text{im } F_i|_{\mathcal{N}_i}$ for all $2 \leq i < j \leq s$.

Let us prove that $B = \{v_1, v_2, \dots, v_s\}$ is a linearly independent set. Since B' is linearly independent we must show that $v_1 \notin kB'$. If $v_1 \in kB'$ then there exist $a_j \in k$, not all of them zero, such that $v_1 = \sum_{j=2}^s a_j v_j$. Let $j_0 = \max\{j : a_j \neq 0\}$ and let $N \in \mathcal{N}_{j_0} \subset \mathcal{N}'$ be a non-zero operator. If we

apply N to both sides of $v_1 = \sum_{j=2}^s a_j v_j$ we obtain zero on the left hand side (since $N \in \mathcal{N}'$), but from conditions (1') and (2'), we obtain $a_{j_0} N(v_{j_0}) \neq 0$ on the right hand side, which is a contradiction.

Let \mathcal{N}_1 be a direct complement of \mathcal{N}' in \mathcal{N} . It is clear that $\mathcal{N} = \mathcal{N}_1 \oplus \dots \oplus \mathcal{N}_s$, $\mathcal{N}_i \neq 0$ for all i , and that conditions (1) and (2) are verified. In order to finish the inductive step we must prove that conditions (3) is verified. In fact we only need to show that $N'v' \in \text{im } F_1|_{\mathcal{N}_1}$ for all $N' \in \mathcal{N}'$ and $v' \in V$.

Since $\dim \mathcal{N} = \dim \ker F_1 + \dim \text{im } F_1 = \dim \mathcal{N}' + r$, we know that $\dim \mathcal{N}_1 = r$. Let $\{\tilde{N}_1, \dots, \tilde{N}_r\}$ be a basis of \mathcal{N}_1 . Given arbitrary $N' \in \mathcal{N}'$ and $v' \in V$ we must show that $N'v' \in \text{span}\{\tilde{N}_1 v_1, \dots, \tilde{N}_r v_1\}$ which in turns is equivalent to prove that $\{N'v', \tilde{N}_1 v_1, \dots, \tilde{N}_r v_1\}$ is linearly dependent. By the definition of v_1 , the set $\{N'(v_1 + tv'), \tilde{N}_1(v_1 + tv'), \dots, \tilde{N}_r(v_1 + tv')\}$ is linearly dependent for all $t \in \mathbb{k}$. Since $N'v_1 = 0$, we have that

$$B' = \{N'v', \tilde{N}_1(v_1 + tv'), \dots, \tilde{N}_r(v_1 + tv')\}$$

is linearly dependent for all $t \neq 0$, and therefore it is linearly dependent for $t = 0$, as we wanted to prove. \square

We are now ready to prove the main result of this section.

Proof of Theorem 4.1. Let (π, V) be a faithful representation of $\bigoplus_{l=1}^k \mathfrak{h}_{m, t^{d_l}}$. By Theorem 2.4 we may assume that π is a nilrepresentation.

We apply Theorem 4.4 to the subspace $\mathcal{N} = \pi(\mathfrak{z})$. We obtain a linearly independent set $B = \{v_1, \dots, v_s\} \subset V$ and decomposition $\mathcal{N} = \mathcal{N}_1 \oplus \dots \oplus \mathcal{N}_s$, $\mathcal{N}_i \neq 0$ for all i , such that the applications $F_i : \mathcal{N} \rightarrow V$ defined by $F_i(N) = N(v_i)$ satisfy

- (1) $F_i|_{\mathcal{N}_i}$ is injective for all $i = 1 \dots s$;
- (2) $\mathcal{N}_j \subset \ker F_i$ for all $1 \leq i < j \leq s$;
- (3) $\mathcal{N}_j V \subset \text{im } F_i|_{\mathcal{N}_i}$ for all $1 \leq i < j \leq s$.

We additionally require that $\pi(Z_l^{d_l-1})v_1 \neq 0$ for all $l = 1, \dots, r$.

Let ϕ be the linear map

$$\phi : \bigoplus_{l=1}^r \mathfrak{h}_{m, t^l} \rightarrow V, \quad \phi(X) = \pi(X)v_1.$$

Note that $\phi|_{\mathfrak{z}} = F_1 \circ \pi$. We claim that

- (i) $\dim \text{im } \phi + \dim \ker F_1 \geq (m+1)d$.
- (ii) $\text{im } \phi \cap \ker F_1 = 0$, and thus $\dim V \geq s + \dim \text{im } \phi$.
- (iii) $d \leq s \dim \text{im } F_1$ and thus $\lceil 2\sqrt{d} \rceil \leq s + \dim \text{im } F_1$.

Proof of (i). It is clear that $\ker \phi$ is a subalgebra of $\bigoplus_{l=1}^k \mathfrak{h}_{m, t^l}$ such that $Z_l^{d_l-1} \notin \ker \phi$. Since $\pi((\ker \phi) \cap \mathfrak{z}) = \ker F_1$, we obtain from Lemma 4.3 that

$$\dim \ker \phi \leq md + \dim \ker F_1.$$

Since $\dim \ker \phi + \dim \text{im } \phi = (2m+1)d$, we obtain part (i).

Proof of (ii). Let $v \in \text{im } \phi \cap \mathfrak{k}B$. Since $v \in \text{im } \phi$ there exists $X \in \bigoplus_{l=1}^r \mathfrak{h}_{m,l}$ such that $\pi(X)(v_1) = v$ and thus there exist $a_1, \dots, a_s \in \mathfrak{k}$ such that

$$(4.2) \quad \pi(X)v_1 = \sum_{i=1}^s a_i v_i.$$

We must prove that $a_i = 0$ for all i . Assume that $a_i \neq 0$ for some i and let $i_0 = \max\{i : a_i \neq 0\}$. Since $\pi(X)$ is a nilpotent endomorphism on V , its only eigenvalue is zero and thus $i_0 > 1$. Now let $N \in \mathcal{N}_{i_0}$ be a non-zero operator and let us apply N to both sides of equation (4.2). We obtain zero on the left hand side, since $N \in \pi(\mathfrak{z})$ and $i_0 > 1$. But from conditions (1) and (2), we obtain $a_{i_0}N(v_{i_0}) \neq 0$ on the right hand side, which is a contradiction.

Proof of (iii). Part (1) and (3) combined imply that $\dim \mathcal{N}_x \geq \dim \mathcal{N}_y$ if $x < y$. In particular $\dim \mathcal{N}_1 \geq \dim \mathcal{N}_j$ for all $j = 1, \dots, s$ and thus

$$d = \dim \mathcal{N} = \sum_{j=1}^s \dim \mathcal{N}_j \leq s \dim \mathcal{N}_1 = s \dim \text{im } F_1.$$

Since $\min\{a + b : a, b \in \mathbb{N} \text{ and } ab \geq d\} = \lceil 2\sqrt{d} \rceil$ for all $d \in \mathbb{N}$, we obtain part (iii).

From part (i) and (ii) it follows that

$$\dim V + \dim \ker F_1 \geq (m+1)d + s,$$

and combining it with part (iii) we obtain

$$\dim V + \dim \ker F_1 + \dim \text{im } F_1 \geq (m+1)d + \lceil 2\sqrt{d} \rceil.$$

Finally, since $\dim \ker F_1 + \dim \text{im } F_1 = d$ we obtain

$$\dim V \geq md + \lceil 2\sqrt{d} \rceil$$

as we wanted to prove. \square

We close this section with the following corollary that is equivalent to the Schur's Theorem mentioned in the introduction.

Corollary 4.6. *Let V be a finite dimensional vector space and let \mathcal{N} be a non-zero abelian subspace of $\text{End}(V)$ consisting of nilpotent operators. Then*

$$\dim V \geq \lceil 2\sqrt{\dim \mathcal{N}} \rceil.$$

Proof. Let B, \mathcal{N}_1, s and F_1 as in Theorem 4.4. Then, by the same argument used in the proof of Theorem 4.1 in its items (ii) and (iii) we obtain that

- (ii)' $\text{im } F_1 \cap \mathfrak{k}B = 0$, and thus $\dim V \geq s + \dim \text{im } F_1$, and
- (iii)' $\dim \mathcal{N} \leq s \dim \text{im } F_1$ and thus $\lceil 2\sqrt{d} \rceil \leq s + \dim \text{im } F_1$.

Therefore $\dim V \geq \lceil 2\sqrt{\dim \mathcal{N}} \rceil$. \square

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